Introduction to Physical Cosmology

Inspired by Professor Andrea Ferrara's lectures

Andrea Ferrara, Luca Marchetti, Marta Monelli

Contents

Contents **iii**

Preface **x**

I Basics **1**

1 Big Bang theory at a glance **3**
1.1 Foundations of Big Bang theory 4
1.1.1 Olbers' paradox 4
1.1.2 The Cosmological Principle 5
1.1.3 Common Big Bang misconceptions 8
1.2 Big Bang pillars 10
1.2.1 Cosmic expansion and Hubble's law 10
1.2.2 Cosmic Microwave Background 14
1.2.3 Abundance of light elements 18
1.3 A rough timeline 20

2 FRW spacetime **27**
2.1 Choosing the coordinate system 30
2.1.1 Spacetime slicing 31
2.1.2 Comoving coordinate system 35
2.2 FRW metric 37
2.2.1 Constraints on the spatial metric 37
2.2.2 Conformal form 40
2.2.3 Spatial slices geometry 42
2.2.4 FRW connection and Ricci tensor 45
2.3 Physical features of a FRW universe 47
2.3.1 Geodesics 47
2.3.2 Cosmological redshift 49
2.3.3 Horizons 53
2.3.4 Distances in Cosmology 58

3 Friedmann equations **63**
3.1 Energy-momentum tensor 64

3.1.1 Energy-momentum tensor . . . 65
3.1.2 Conservation equations . . . 66
3.2 Friedmann equations . . . 68
3.2.1 Newtonian derivation . . . 69
3.2.2 Density parameters . . . 72
3.3 Cosmic inventory . . . 73
3.3.1 Radiation . . . 76
3.3.2 Matter . . . 80
3.3.3 Dark Matter . . . 81
3.3.4 Dark energy . . . 88
3.3.5 Cosmic dynamics and density parameters . . . 96

4 Cosmic dynamics 99
4.1 Scale factor evolution . . . 99
4.1.1 General behaviour of the Universe . . . 99
4.1.2 Analytic cosmological models . . . 102
4.2 Beyond the scale factor evolution . . . 107
4.2.1 Density parameter evolution . . . 107
4.3 Times and distances . . . 109
4.3.1 Times . . . 109
4.3.2 Distances . . . 111

5 Cosmic thermodynamics 117
5.1 Equilibrium thermodynamics . . . 118
5.1.1 Distribution function . . . 118
5.1.2 Equilibrium in a FRW Universe . . . 121
5.1.3 Equilibrium thermodynamics . . . 129
5.2 Decoupling and freeze-out . . . 135
5.2.1 Decoupled distribution functions . . . 135
5.2.2 The freeze-out of stable particles . . . 138
5.2.3 Hot and cold relics . . . 142
5.2.4 Two examples of cold relics . . . 146
5.3 Decaying particles . . . 150
5.3.1 Estimating the entropy production . . . 150
5.3.2 A more detailed analysis . . . 151

6 The early Universe 157
6.1 Lepton epoch . . . 158
6.1.1 Neutrino decoupling . . . 159
6.2 Cosmological nucleosynthesis . . . 162
6.2.1 Preliminaries . . . 162
6.2.2 Deuterium bottleneck . . . 168
6.2.3 Light elements production . . . 170
6.2.4 BBN as a probe of BSM physics . . . 174
6.3 The plasma era . . . 174
6.3.1 Radiation-matter equality . . . 175

6.3.2 Baryon-photon equilibrium 178
6.3.3 Recombination epoch 180
6.3.4 Photon decoupling 187
6.4 After photon decoupling 192

7 Large scale structures **195**
7.1 The perturbation theory toolkit 196
7.1.1 Super- and sub-horizon modes 198
7.1.2 Statistical description 202
7.1.3 The filtered overdensity field 204
7.2 Linear perturbation theory 205
7.2.1 Non-expanding background 207
7.2.2 Expanding background 210
7.2.3 Evolution during RD, MD and ΛD 212
7.2.4 Growth factor and transfer function 214
7.2.5 Linear power spectrum 216
7.3 Dark Matter halos 217
7.3.1 Non-linear collapse: spherical model 218
7.3.2 Reaching equilibrium 223
7.3.3 Virialized systems 224
7.3.4 Press-Schechter mass function 227
7.4 Theory vs observations 231
7.4.1 Dark matter clustering 233
7.4.2 Galaxy clustering 238
7.4.3 Numerical methods 244

8 CMB anisotropies **245**
8.1 Describing anisotropies 246
8.1.1 Spherical harmonics decomposition 247
8.1.2 Anisotropies from inhomogeneities 251
8.2 Inhomogeneities at decoupling 254
8.2.1 Continuity and Euler equations 256
8.2.2 Acoustic oscillations 258
8.3 Impact on the C_ℓ coefficients 258
8.3.1 Towards a more refined picture 263
8.4 Polarization 264
8.4.1 Describing polarization 267
8.4.2 Angular power spectra 278

II Advanced Topics **281**

9 The inflationary epoch **283**
9.1 Shortcomings of standard Cosmology 283
9.1.1 The Cosmic Cauchy problem 283
9.1.2 The horizon and flatness problems 286

9.2 The inflationary paradigm 290
9.2.1 The shrinking Hubble radius 290
9.2.2 Conditions for inflation 292
9.2.3 Conformal diagram of inflation 294
9.3 'Old' inflationary models 295
9.3.1 De Sitter inflation 295
9.3.2 Guth's inflation 297
9.4 Slow-roll inflation 300
9.4.1 General single field slow-roll 301
9.4.2 Small and large field slow-roll 303
9.5 Beyond slow-roll 311
9.5.1 Hybrid inflation 311
9.5.2 Modified gravity and non-minimal coupling 315
9.5.3 K-inflation 319
9.5.4 Effective field theory of inflation 323
9.6 Reheating 326

10 Relativistic perturbation theory **331**
10.1 Choosing an appropriate variable set 332
10.1.1 Perturbed spacetime 334
10.1.2 Perturbed matter 336
10.2 The gauge problem 338
10.2.1 Gauge-invariant metric perturbations 340
10.2.2 Gauge-invariant matter perturbations 342
10.2.3 Gauge-invariant curvature perturbations 344
10.2.4 From gauge to gauge 347
10.3 Equations in helicity sectors 349
10.3.1 Scalar perturbations 350
10.3.2 Vector and tensor perturbations 355

11 Evolution of perturbations **359**
11.1 Evolution of V-T perturbations 360
11.1.1 Vector perturbations 360
11.1.2 Tensor perturbations 361
11.2 Initial data for scalar modes 367
11.2.1 Initial conditions deep in RD 368
11.2.2 Adiabatic and isocurvature modes 369
11.2.3 The variables ζ and $\mathcal{R}$ 374
11.3 Evolution of scalar modes 376
11.3.1 Overview 377
11.3.2 During radiation domination 382
11.3.3 Early MD (before recombination) 387
11.3.4 After recombination 393
11.3.5 Transfer and growth functions 398

13 Physics of CMB **401**

13.1 Observed anisotropies 402
13.1.1 The line of sight integral 405
13.1.2 SW, ISW and Doppler terms 409
13.2 Effective temperature at $\eta_{\rm dec}$ 410
13.2.1 Acoustic oscillations 411
13.2.2 Damping effects 415
13.2.3 Momentum power spectrum 420
13.3 Angular power spectrum 423
13.3.1 The observed C_ℓ 423
13.3.2 Large scales 431
13.3.3 Intermediate scales 434
13.3.4 Small scales 440
13.4 Cosmological parameters 441
13.4.1 Curvature 441
13.4.2 Minimal ΛCDM parameters 444

14 Initial data from inflation **453**
14.1 Quantum fields on curved space 454
14.1.1 Field quantization on flat space 454
14.1.2 Field quantization on curved space 461
14.1.3 Quantum fields on FRW space 466
14.1.4 From quantum to classical 471
14.2 Primordial perturbations 475
14.2.1 Overview 475
14.2.2 The Mukhanov-Sasaki equation 477
14.2.3 Scalar perturbations in slow-roll 480
14.2.4 Tensor perturbations 485

Appendices **491**

A Differential geometry **493**
A.1 Differential manifolds 494
A.2 Manifolds and tensor fields 497
A.2.1 The tangent space 497
A.2.2 Tensors 499
A.2.3 Vector and tensor fields 502
A.3 Differential forms 502
A.4 Covariant derivatives 504
A.5 Parallel transport and geodesics 506
A.5.1 Geodesics 507
A.6 Curvature 508

B Riemannian geometry **511**
B.1 The metric tensor 511
B.2 Geodesics revisited 515

B.2.1 Geodesics as extremal curves . . . 515
B.2.2 Levi-Civita connection . . . 517
B.2.3 Riemann normal coordinates . . . 518
B.2.4 Geodesic deviation . . . 520
B.3 Riemannian curvature . . . 521
B.3.1 Metric constancy and flatness . . . 523
B.4 Integration . . . 525
B.4.1 Tensor densities . . . 526
B.4.2 Volume and surface elements . . . 527
B.4.3 Volume elements: momentum space . . . 528
B.5 Congruence of timelike geodesics . . . 531
B.5.1 Vorticity and Frobenius' Theorem . . . 532
B.5.2 Expansion: Raychaudhuri equation . . . 533
B.6 Maximally symmetric spaces . . . 536

C Basics of General Relativity 541
C.1 Gravity and geometry . . . 541
C.1.1 Newtonian gravity . . . 541
C.1.2 Weak Equivalence Principle . . . 542
C.1.3 Einstein's Equivalence Principle . . . 544
C.1.4 EPs and metric theories of gravity . . . 546
C.1.5 Principle of General Covariance . . . 547
C.2 Distances and time intervals . . . 549
C.3 Einstein equations . . . 552
C.3.1 Building of the total action . . . 552
C.3.2 Einstein equations . . . 556
C.4 Metric energy-momentum tensor . . . 559
C.4.1 Canonical EMT . . . 559
C.4.2 Metric and canonical EMTs . . . 560
C.4.3 Simple examples . . . 561

D Classical physics & gravity 567
D.1 Relativistic fluidodynamics . . . 567
D.1.1 Thermodynamic properties . . . 567
D.1.2 Kinematic properties . . . 568
D.1.3 Hydrodynamics of a perfect fluid . . . 569
D.1.4 Thermodynamics of a simple fluid . . . 572
D.2 Relativistic kinetic theory . . . 575
D.2.1 One-particle distribution function . . . 576
D.2.2 The Boltzmann equation . . . 579
D.3 Boltzmann equation in Cosmology . . . 583
D.3.1 Boltzmann equation for photons . . . 584
D.3.2 Boltzmann equation for matter . . . 589
D.3.3 Newtonian limit . . . 592

E Gaussian random fields 597

E.1 Random fields: useful tools 598
E.2 Power spectrum 600

Bibliography **603**

Preface

This book aims at providing Physics graduate and PhD students with an introduction to Physical Cosmology. It may also serve as a reference and entry point for some more specific aspects of modern Cosmology for more experienced researchers. Physical Cosmology is concerned with the study of the largest-scale structures and dynamics of the Universe and investigates fundamental questions about its origin, structure, evolution, and ultimate fate. The key feature of this discipline is the strong interconnection between theoretical approaches and the guidance provided by data collected using a variety of observational techniques in different bands of the electromagnetic spectrum.

Therefore, it does not come as a surprise that its beginnings date back by about one century, when A. Einstein developed the theory of General Relativity, describing the geometrical and dynamical properties of the Universe, and immediately after E. Hubble experimentally discovered cosmic expansion. The synergy between theory and observations has been a constant and fruitful feature of the dazzling growth and success that has characterized Physical Cosmology in the last century, and still continues in a vigorous manner at present. Such path has been studded with many Nobel Prizes, the most recent (but probably not the last) of which has been awarded to Prof. J. Peebles, who is unanimously recognised as the father of the most modern approach to Physical Cosmology adopted by this book.

Physical Cosmology is one of the fastest growing areas in Physics, in terms of new experimental discoveries, theories and numerical simulations now allowed by advanced telescopes, experiments and supercomputers. Being equipped with a solid and updated physical background is therefore a pre-requisite to carry out successful research in the field and to exploit at best these unique opportunities.

The presented material is largely based on the graduate and PhD course Lectures I have been giving at Scuola Normale Superiore in the last decade. However, it is only thanks to the wonderful and outstanding help of my two former students, Luca Marchetti and Marta Monelli, who these materials have properly organized and extended in the form presented in the book. Ideally, no specific previous knowledge of General Relativity or Field Theory is necessary to read the various Chapters. We thus believe that every Physics graduate student can grasp the explanations and the concepts contained therein.

We have also conceived the book as a useful tool for teachers to organize their graduate course in Physical Cosmology, although we are aware that there is probably more material in the book that any single one-semester course can contain. For this reason the book is divided in two parts: the first contains more standard material that is covered in many commonly available cosmology textbooks. It is meant to provide a solid background from a modern perspective to scholars entering the field for the first time. The second part deals with more advanced topics that are usually not part of textbooks devoted to Physical Cosmology. As such they are more suitable for PhD courses or advanced researchers. These include a simplified but yet rigorous treatment of inflation and relativistic perturbation theory, a detailed description of CMB physics, and an introduction to quantum field theory in curved spaces.

A considerable amount of useful and technical material has been included in the Appendixes. As General Relativity is not a pre-requisite of the course that forms the basis of this book, we have deemed as useful to provide the reader with a concise summary of the key tools and concepts of Einstein's theory. Therefore, there are Appendixes dealing with differential (App. A) and Riemannian (App. B) geometry necessary to penetrate the key concepts of General Relativity (App. C). In addition, App. D and App. E represent a handy primer on relativistic fluid dynamics and gaussian random fields.

Finally, we made a special effort in each Chapter to clearly and rigorously (perhaps somewhat tediously, one might argue!) expand derivations or particularly complex or subtle concepts. Besides more traditional margin notes, we use three different kinds of boxes: boxes in the margins are used to show short derivations of equations presented in the main text; full-page boxes marked with the symbol contain additional details about topics that are useful, but not necessary, in order to obtain a better understanding of the matter discussed in the main text; full-page boxes marked with at the end of each chapter provide a guide among the references which the chapter itself is based on and list some further readings. The content of these boxes could be skipped in a first approach or by more advanced readers.

I hope that students can enjoy reading and studying this book, appreciate the beauty of the Universe through it, and have at least as much fun as Luca, Marta and myself had writing it.

Andrea Ferrara